Optimal Nonplanar Escape from Circular Orbits

T. N. EDELBAUM*
Massachusetts Institute of Technology, Cambridge, Mass.

An approximate analytical solution is obtained for minimum impulse transfer between a given circular orbit and a given hyperbolic velocity vector at infinity. The transfer time between the first and last impulses is assumed fixed and much larger than the period of the circular orbit. A minimum allowable periapsis radius is prescribed. The solution is developed at the initial terms in an asymptotic expansion in inverse powers of the transfer time. Below a critical inclination the optimal solution requires three impulses, while above this inclination four impulses are required.

Introduction

THE general problem of transfer between an elliptic orbit and a hyperbolic asymptote has aroused considerable interest in recent years.[1-10] Gerbracht and Penzo[1] state that it is almost certainly the most important and potentially the most promising trajectory optimization problem. The present paper is concerned with an important case of this general problem, where the elliptic orbit is circular.

A good summary of previous work on both the circular and elliptic problems may be found in.[2] Single impulse transfer from a circular orbit is discussed in.[3-5] Two impulse transfer is discussed in.[4-5] The optimal one and two impulse transfers are derived in.[4] Suboptimal three impulse transfers are discussed in.[1-6] Numerical examples of optimum fixed time multiple impulse trajectories are given in.[7-8] The basic idea of the absolute optimal transfer which requires infinite time is discussed in.[9]

The fuel requirements of the one and two impulse transfer become prohibitively large unless the inclination of the hyperbolic asymptote to the circular orbit plane is quite small. These fuel requirements can be reduced to reasonable levels for any inclination if sufficient time is allowed for the transfer. The present paper is intended to provide an approximate analytic solution to this problem for large transfer times.

The transfer time will be defined to be the time between the first and last impulses. The solution is developed as an asymptotic expansion in powers of this transfer time to the minus one third power. The zero-order term in the asymptotic expansion for the required impulse is that for the absolute minimum solution requiring infinite time. This term is the sum of two finite impulses and one or more infinitesimal impulses at infinity.[10]

The first impulse is the difference between circular and escape velocity in the initial circular orbit

$$\lim_{t\to\infty} \Delta V_1 = (2\mu/R_0)^{1/2} - (\mu/R_0)^{1/2} \qquad (1)$$

The last impulse is the difference between the hyperbolic periapsis velocity and escape velocity at the minimum allowable periapsis radius.

$$\lim_{t\to\infty} \Delta V_f = [(2\mu/R_p) + V_\infty^2]^{1/2} - (2\mu/R_p)^{1/2} \qquad (2)$$

The minimum allowable periapsis radius is assumed to be less than or equal to the radius of the circular orbit.

Presented as Paper 70-1038 at the AAS/AIAA Astrodynamics Conference, Santa Barbara, Calif., August 19–21, 1970; submitted September 18, 1970; revision received June 24, 1971. This work was supported by NASA Contracts NAS9-9024 with MIT and NAS12-656 with AMA. The author would like to thank W. C. Bean of the Manned Spacecraft Center for stimulating his interest in this problem and for making available his numerical results.

* Deputy Associate Director, Charles Stark Draper Laboratory. Associate Fellow AIAA.

The optimal transfer will require either three or four impulses depending upon the eccentricity (e) of the final escape hyperbola and the minimum angle between the hyperbolic asymptote and the plane of the circular orbit (θ). The final escape hyperbola will always have its periapsis at the minimum allowable radius. Its eccentricity is given by Eq. (3);

$$e = 1 + (R_p V_\infty^2/\mu) \qquad (3)$$

The limiting value of true anomaly for this hyperbola is given by Eq. (4)

$$\lim_{R\to\infty} f = \cos^{-1}(-1/e) \equiv f_\infty, \quad \pi/2 \le f_\infty \le \pi \qquad (4)$$

If the direction of the hyperbolic asymptote is assumed to pierce the celestial sphere at the north pole, the locus of allowable periapsides will correspond to a small circule of latitude in the southern hemisphere. The optimal transfer will be a three impulse transfer if the plane of the circular orbit cuts this small circle ($\theta \le \pi - f_\infty$) and will be a four impulse transfer if it does not cut the small circle ($\pi/2 \ge \vartheta > \pi - f_\infty$).

The zero-order term in the asympotic expansion of the required total impulse is given by the sum of Eqs. (1) and (2) for both the three and four impulse transfers. The order of the next term depends upon whether the transfer uses three or four impulses. For the four impulse case, the next term is of the order of transfer time to the minus one third power. For the three impulse case, the next term is of the order of transfer time to the minus two thirds power. It is necessary to use much longer times for the four impulse case than for the three impulse case to obtain close to the absolute minimum total impulse.

Three Impulse Transfer

The projection of the circular orbit plane on the celestial sphere will cut the periapsides locus of the escape hyperbola in two points. The position of the first and third impulses on the celestial sphere is close to the point where the velocity in the circular orbit is directed away from the periapsides locus, Fig. 1. The position of the second impulse on the celestial

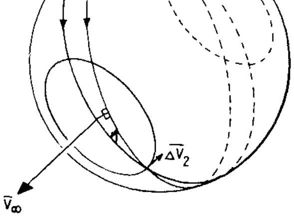

Fig. 1 Three impulse transfer.

sphere is almost opposite the position of the other two impulses.

The original orbit plane will have to be rotated through an angle i into a plane specified by the location of the third impulse and the hyperbolic asymptote. This angle i is given by spherical trigonometry as Eq. (5).

$$\sin i = \sin\theta/\sin f_\infty = e\sin\theta/(e^2-1)^{1/2},\ 0 \le i \le \pi/2 \quad (5)$$

The first impulse is usually located slightly outside the periapsides locus. Its angular distance from that locus is given by Eq. (6)

$$\omega_1 = 2\frac{R_p}{R_a}\left[\frac{R_0(e^2-1)\sin^4 i}{R_0+R_p-2(R_0R_p)^{1/2}\cos i}\right]^{1/2} \times \left\{\frac{3e-1-(2e+2)^{1/2}}{e} + \frac{R_p^{1/2}-R_0^{1/2}\cos i}{[R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}}\right\} + 0(R_a^{-2}) \quad (6)$$

The quantity R_a in Eq. (6) is the apoapsis radius of the first and second transfer ellipses, they are essentially the same. The solution is developed as an asymptotic expansion in this quantity.

The first impulse is used to inject the vehicle onto a highly eccentric ellipse with the same periapsis radius as the circular orbit. Its magnitude is given by Eq. (7).

$$\Delta V_1 = (2\mu/R_0)^{1/2}[1 - \tfrac{1}{2} - \tfrac{1}{2}(R_0/R_a)] + 0(R_a^{-2}) \quad (7)$$

This impulse produces a small part of the plane change. It is inclined to the original orbit plane by the angle β_1 given by Eq. (8).

$$\beta_1 = (R_0/R_a)\{2R_p/[R_0+R_p-2(R_0R_p)^{1/2}\cos i]\}^{1/2} \times \sin i + 0(R_a^{-2}) \quad (8)$$

This first impulse has a negligible radial component of $0(R_a^{-11/2})$. It rotates the orbit plane through an angle i_1 given by Eq. (9).

$$i_1 = \frac{R_0}{R_a}\frac{(2^{1/2}-1)R_p^{1/2}\sin i}{[R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}} + 0(R_a^{-2}) \quad (9)$$

The second impulse is applied slightly beyond apoapsis of the first transfer ellipse. Its true anomaly is given by Eq. (10)

$$f_{21} = \pi + 2\frac{R_0}{R_a}\frac{R_p(e^2-1)^{1/2}\sin^2 i\cos i}{[R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}} + 0(R_a^{-2}) \quad (10)$$

This impulse occurs slightly outside the projection of the periapsides locus into the other hemisphere. The second impulse is used to accomplish most of the plane change and to lower the periapsis to within $0(R_a^{-2})$ of R_p. Its magnitude is given by Eq. (11).

$$\Delta V_2 = \frac{\{2\mu[R_0+R_p-2(R_0R_p)^{1/2}\cos i]\}^{1/2}}{R_a} + 0(R_a^{-2}) \quad (11)$$

This impulse has a small radial component directed downwards and a large out of plane component. The angle with the local horizontal is given by Eq. (12) while the angle with the orbit plane of the first transfer ellipse is given by Eq. (13)

$$\alpha_{21} = \frac{R_0}{R_a}\frac{(R_p^{1/2}\cos i - R_0^{1/2})R_p(e^2-1)^{1/2}\sin^2 i\cos i}{[R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{3/2}} + 0(R_a^{-2}) \quad (12)$$

$$\sin\beta_{21} = -\left[\frac{R_p\sin^2 i}{R_0+R_p-2(R_0R_p)^{1/2}\cos i}\right]^{1/2} \times \left\{1 - \frac{R_0}{R_a}\left[1 - \frac{R_p(e^2-1)\sin^2 i\cos^2 i}{R_0+R_p-(R_0R_p)^{1/2}\cos i} \times \left(2 - \frac{R_p\sin^2 i}{R_0+R_p-(R_0R_p)^{1/2}\cos i}\right)\right]\right\} + 0(R_a^{-2}) \quad (13)$$

The true anomaly after entry onto the second transfer ellipse is given by Eq. (14).

$$f_{22} = \pi + 2\frac{R_p}{R_a}\frac{(R_0R_p)^{1/2}(e^2-1)^{1/2}\sin^2 i\cos i}{R_0+R_p-(R_0R_p)^{1/2}\cos i} + 0(R_a^{-2}) \quad (14)$$

An internal check on the consistency of the results was satisfied by noting that Eqs. (10) and (14) both correspond to local maxima of the primer vector and to the same radius.

The third impulse is given inside the periapsides locus. Its true anomaly on the second transfer ellipse is given by Eq. (15).

$$f_{32} = -8\frac{R_p}{R_a}\frac{R_0^{1/2}(e^2-1)^{1/2}\sin^2 i\cos i}{[R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}} + 0(R_a^{-2}) \quad (15)$$

The magnitude of the third impulse is given by Eq. (16).

$$\Delta V_3 = [(2\mu/R_p) + V_\infty^2]^{1/2} - (2\mu/R_p)^{1/2}[1 - \tfrac{1}{2}(R_p/R_a)] + 0(R_a^{-2}) \quad (16)$$

This impulse has a small downward component and a small out of plane component. The angle with the local horizontal and the plane of the second transfer ellipse are given by Eqs. (17) and (18).

$$\alpha_{32} = f_{32}/4 + 0(R_a^{-2}) \quad (17)$$

$$\beta_{32} = (R_p/R_a)\{R_0^{1/2}\sin i/[R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}\} \quad (18)$$

The inclination change produced by this impulse is given by Eq. (19).

$$i_3 = \{1 - [2/(e+1)]^{1/2}\}\beta_{32} + 0(R_a^{-2}) \quad (19)$$

The final hyperbola is entered before its periapsis at a true anomaly given by Eq. (20).

$$f_{33} = [(e+1+(2e+2)^{1/2})/4e]f_{32} + 0(R_a^{-2}) \quad (20)$$

The total time for the maneuver is given by Eq. (21).

$$t_{13} = \pi(R_a^3/2\mu)^{1/2}[1 + \tfrac{3}{4}(R_0/R_a) + \tfrac{3}{4}(R_p/R_a) + 0(R_a^{-2})] \quad (21)$$

The total ΔV for the maneuver is given by Eqs. (22) and (23).

$$\Sigma\Delta V = \Delta V_{t=\infty} + (2\mu)^{1/2}/R_a\{(R_p^{1/2}/2) - (R_0^{1/2}/2) + [R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}\} + 0(R_a^{-2}) \quad (22)$$

$$\Sigma\Delta V = \Delta V_{t=\infty} + (2\mu)^{1/6}\pi^{2/3}\{(R_p^{1/2}/2) - (R_0^{1/2}/2) + [R_0+R_p-2(R_0R_p)^{1/2}\cos i]^{1/2}\}t_{13}^{-2/3} + 0(t_{13}^{-4/3}) \quad (23)$$

The effect of the time constraint on the solution for the primer vector can be estimated from the value of the Hamiltonian H and the Lagrange multiplier for the mean anomaly λ_M, Ref. 11

$$H = (a^3/\mu)^{1/2}\lambda_M = -\partial\Sigma\Delta V/\partial t_{13} = 0(t_{13}^{-5/3}) = 0(r_a^{-5/2}) \quad (24)$$

For the time open problem both H and λ_M will be zero. For the time fixed problem they will introduce radial components $\delta\lambda_R$ into the primer vector. The order of these radial components can be estimated from Eq. (24).

$$\delta\lambda_{R_1} = (a/\mu)^{1/2}\lambda_M[(1-e)/e] = 0(r_a^{-11/2}) \quad (25)$$

$$\delta\lambda_{R_2} = -4(a/\mu)^{1/2}\lambda_M[(1+e)^2/e] = 0(r_a^{-7/2}) \quad (26)$$

$$\delta\lambda_{R_3} = (a/\mu)^{1/2}\lambda_M[(1-e)^2/e] = 0(R_a^{-11/2}) \quad (27)$$

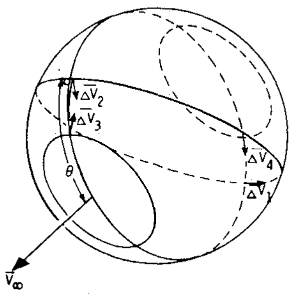

Fig. 2 Four impulse transfer.

All of these radial components of the primer vector are of high enough order to be neglected.

Four Impulse Transfer

In the four-impulse case it is no longer possible to use a coaxial transfer where the highly eccentric ellipses are rotated around their common major axis. In the four impulse case, the major axis itself must be changed in orientation. The rotation of the major axis requires a larger ΔV of order $R_a^{-1/2}$ or $t^{-1/3}$. To a first approximation, the minimum ΔV four impulse transfer minimizes the rotation of the major axis.

The minimum possible rotation of the major axis is through an angle 2ϕ defined by Eq. (28)

$$2\phi = \theta + f_\infty - \pi \tag{28}$$

This angle is the minimum angle between the plane of the circular orbit and the periapsides locus. It falls within the limits given by Eq. (29)

$$0 < 2\phi \leq f_\infty - \pi/2 \leq \pi/2 \tag{29}$$

In the following analysis, both ϕ and f_∞ will be assumed to be of order unity. The cases where either or both of these angles are small requires separate investigation.

The first of the four impulses is a tangential impulse which is used to leave the initial circular orbit.

$$\Delta V_1 = (2\mu/R_0)^{1/2}[1 - 1/2^{1/2} - \tfrac{1}{2}(R_0/R_a)] + 0(R_a^{-2}) \tag{30}$$

This impulse is applied at a point almost opposite the point of closest approach of the trace of the circular orbit and the periapsides locus on the celestial sphere, Fig. 2. It is applied beyond the diametrically opposite point by an angle f_1 given by Eq. (31).

$$f_1 = (R_0/R_a)^{1/2}[(1 + 2\cos\phi)/(1 + \sin^2\phi)^{1/2}] + 0(R_a^{-1}) \tag{31}$$

This angle positions the second impulse at the optimal location on the celestial sphere.

The second impulse is applied at a radius given by Eq. (32).

$$R_2 = R_a[(1 + \sin^2\phi)/2 + 0(R_a^{-1})] \tag{32}$$

The magnitude of the second impulse is given by Eq. (33).

$$\Delta V_2 = \left(\frac{2\mu}{R_a}\right)^{1/2} \frac{\sin\phi}{(1 + \sin^2\phi)^{1/2}} - \frac{(2\mu R_p)^{1/2}}{R_a} \times \left(\frac{\cos\phi}{1 + \sin^2\phi}\right)^2 \left[1 + \frac{3e - 1 - (2e + 2)^{1/2}}{4e(1 + \sin^2\phi)}\right] + 0(R_a^{-3/2}) \tag{33}$$

The eccentricity e is that of the final hyperbola; Eq. (3). This second impulse has no radial component and is applied at almost right angles to the plane of the circular orbit and the first transfer ellipse. The angle with the initial plane is given by Eq. (34). Note that this impulse removes half of the circumferential velocity of the first transfer ellipse.

$$\beta_2 = (\pi/2) + (R_0/R_a)^{1/2}[1/\sin\phi(1 + \sin^2\phi)^{1/2}] + 0(R_a^{-1}) \tag{34}$$

The second impulse transfers the vehicle onto an ellipse which is almost at right angles to the plane of the original orbit. This ellipse is entered at its semiminor axis. The vehicle coasts past apoapsis and the third impulse is applied when the other end of the minor axis is reached. The radius will then be the same as given by Eq. (32). The magnitude of the third impulse is given by equation

$$\Delta V_3 = \Delta V_2 + [2(2\mu R_p)^{1/2}/R_a(1 + \sin^2\phi)] + 0(R_a^{-3/2}) \tag{35}$$

The third impulse has no radial component and is directed almost opposite to the circumferential velocity in the second transfer ellipse.

$$\beta_3 = \pi + (R_0/R_a)^{1/2}[1/\sin\phi(1 + \sin^2\phi)^{1/2}] + 0(R_a^{-1}) \tag{36}$$

This impulse changes the orbit plane by almost 180°. It orients the third transfer ellipse so that it is coplanar with the hyperbolic asymptote.

The final impulse is given just before periapsis of the third transfer ellipse. The true anomaly at which this impulse is applied is given by Eq. (37).

$$f_4 = -2(R_p/R_a)^{1/2}[\cos\phi/(1 + \sin^2\phi)^{1/2}] + 0(R_a^{-1}) \tag{37}$$

This impulse is coplanar but has a small radial component directed towards the planet. The angle with the local horizontal is given by Eq. (38).

$$\alpha_4 = f_4/4 \tag{38}$$

The magnitude of the fourth impulse is given by Eq. (39).

$$\Delta V_4 = \left[\left(\frac{2\mu}{R_p}\right) + V_\infty^2\right]^{1/2} - \left(\frac{2\mu}{R_p}\right)^{1/2} + \frac{(2\mu R_p)^{1/2}}{2R_a} \times \left[1 + \frac{\cos^2\phi}{(1 + \sin^2\phi)^3} \frac{3e - 1 - (2e + 2)^{1/2}}{8e}\right] + 0(R_a^{-2}) \tag{39}$$

Following the fourth impulse the vehicle is injected onto the correct escape hyperbola slightly before its periapsis. The time spent on the three transfer ellipses is given by Eqs. (40) and (42).

$$t_{12} = (R_a^3/8\mu)^{1/2}[\pi - \cos^{-1}(\sin^2\phi) - (1 - \sin^4\phi)^{1/2} + 0(R_a^{-1})] \tag{40}$$

$$t_{23} = (R_a^3/8\mu)^{1/2}[(1 + \sin^2\phi)^{3/2}(\pi + 2\cos\phi) + 0(R_a^{-1})] \tag{41}$$

$$t_{34} = t_{12} \tag{42}$$

$$\Sigma\Delta V = \Delta V_{t=\infty} + \left(\frac{2\mu}{R_a}\right)^{1/2} \frac{2\sin\phi}{(1 + \sin^2\phi)^{1/2}} - \frac{(2\mu R_0)^{1/2}}{2R_a} + \frac{(2\mu R_p)^{1/2}}{R_a}\left\{\frac{1}{2} + \frac{2}{1 + \sin^2\phi} - \frac{\cos^2\phi}{(1 + \sin^2\phi)^2} \times \left[2 + \frac{7}{16}\frac{3e - 1 - (2e + 2)^{1/2}}{e(1 + \sin^2\phi)}\right]\right\} + 0(R_a^{-3/2}) \tag{43}$$

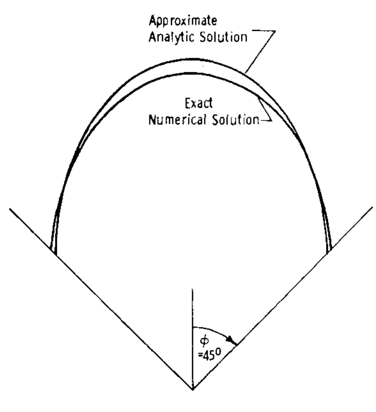

Fig. 3 Transfer between unit eccentricity ellipses ϕ = 45°.

Four Impulse Transfer—Derivation of the Lower-Order Terms

The terms of order $R_a^{-1/2}$ and $t_{14}^{-1/3}$ in the four impulse case correspond to transfer between two equal unit eccentricity ellipses. These are degenerate ellipses having a finite major axis but zero angular momentum and no definable orbit plane. The transfer problem is to change the orientation of the major axis so that the fourth impulse may be placed at an optimal position in space. The departures of these ellipses from unit eccentricity contribute higher-order terms which will be considered in a later section.

The transfer is assumed to start with an infinitesimal impulse at periapsis which produces the optimal apoapsis radius for the given transfer time. A second impulse is used to transfer from the first unit eccentricity ellipse to an intermediate ellipse. A third impulse is then used to transfer onto the second unit eccentricity ellipse. An infinitesimal impulse at periapsis is assumed to end the fixed time maneuver. The maneuver is assumed to be symmetrical.

An approximate analytic solution will be obtained and compared with an exact numerical solution of this problem. The assumption used to obtain the approximate solution is that the radial component of the primer vector is of higher order and may be neglected. For this problem the order of the Hamiltonian may be found from Eq. (44)

$$H = (a^3/\mu)^{1/2}\lambda_M = -(\partial\Sigma\Delta V/\partial t_{14}) = 0(t_{14}^{-4/3}) = 0(R_a^{-3}) \quad (44)$$

On the unit eccentricity ellipse the radial component of the primer vector is given by Eq. (45)

$$\lambda_R = (a/\mu)^{1/2}\lambda_M\{[3E\sin E/(1-\cos E)] - 5 - \cos E\} = 0(R_a^{-3}) \quad (45)$$

This term is of higher order and may be neglected to the order of the rest of the analysis.

The circumferential component of the primer vector on the unit eccentricity ellipse is independent of the magnitude of the Hamiltonian. As a result the fixed-time problem is reduced to a minimum ΔV time-open transfer with circumferential impulses.

The latter problem can be solved by the following procedure. On the intermediate transfer ellipse, the radial and circumferential velocities are given by Eqs. (46) and (47).

$$\dot{R} = (\mu/p)^{1/2}e\sin f \quad (46)$$

$$R\dot{\theta} = (\mu/p)^{1/2}(1 + e\cos f) \quad (47)$$

The semilatus rectum and eccentricity are given by Eqs. (48)

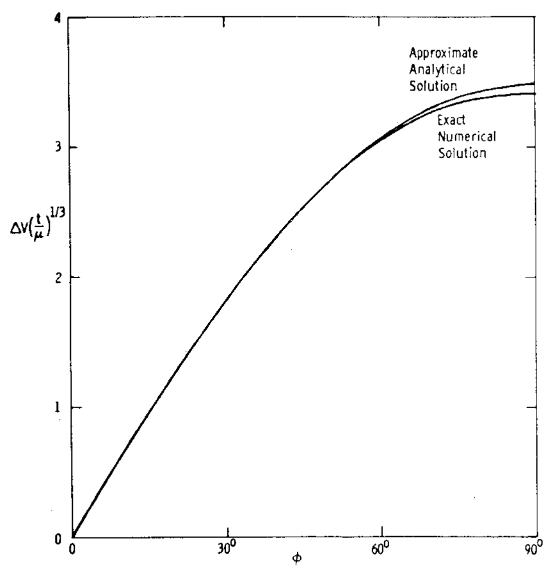

Fig. 4 Cost function vs half rotation angle.

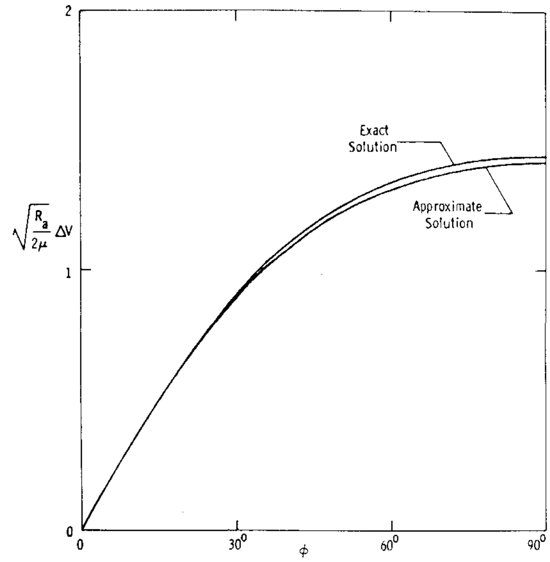

Fig. 5 Total velocity increment vs half rotation angle.

and (49).

$$p = R^4\dot{\theta}^2/\mu \quad (48)$$

$$e = R^2\dot{R}\dot{\theta}/\mu \sin f = (R^3\dot{\theta}^2 - \mu)/\mu \cos f \quad (49)$$

Combining these four equations yields Eq. (50) for the circumferential velocity as a function of radius, radial velocity and true anomaly.

$$R\dot{\theta} = (\dot{R}\cot f/2) + (\dot{R}^2\cot^2 f/4)^{1/2} + (\mu/R) \quad (50)$$

On the unit eccentricity ellipses, the radius and radial velocity are connected by Eq. (51)

$$\mu/R = (\dot{R}^2/2) + (\mu/R_a) \quad (51)$$

As a circumferential impulse will not change radial velocity, Eq. (43) may be substituted into Eq. (50) to yield Eq. (52).

$$R\dot{\theta} = (\dot{R}\cot f/2) + [(\dot{R}^2\cot^2 f/4) + (\dot{R}^2/2) + (\mu/R_a)]^{1/2} \quad (52)$$

This equation may be differentiated with respect to R to find the stationary minimum value of circumferential velocity. The optimum value of R that yields this stationary minimum is given by Eq. (53).

$$\dot{R} = (2\mu/R_a)\cos f/(1 + \sin^2 f) \quad (53)$$

The radius corresponding to this radial velocity is given by Eq. (54).

$$R = R_a[(1 + \sin^2 f)/2] \quad (54)$$

This optimum radius is always between the semiminor axis and the apopapsis and is consistent with the necessary conditions on the primer vector for unit eccentricity ellipses. The true anomaly f will always fall in the second quadrant so that the radial velocity will be positive at entry onto the transfer ellipse. The half-angle of rotation of the major axis is given by Eq. (55) and always lies in the first quadrant

$$\phi = \pi - f \quad (55)$$

By taking careful note of quadrants and signs Eq. (56) for the circumferential velocity may be obtained from Eq. (52).

$$R\dot{\theta} = (2/R_a)^{1/2}[\sin\phi/(1 + \sin^2\phi)^{1/2}] = \Delta V_2 = \Delta V_3 \quad (56)$$

The eccentricity and semimajor axis of the transfer ellipse is given by Eqs. (57) and (58).

$$e = \cos\phi \quad (57)$$

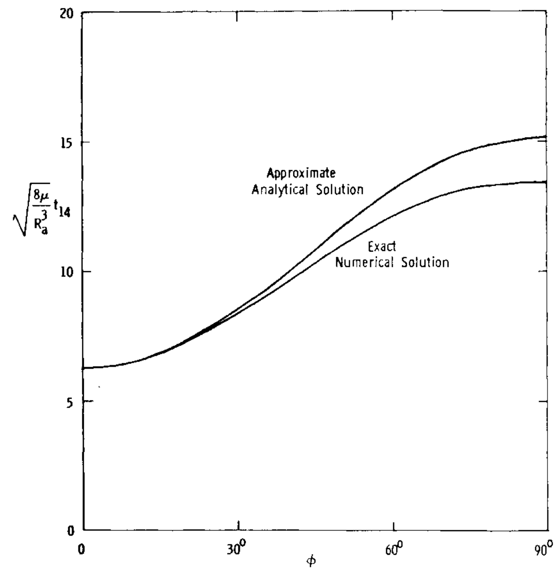

Fig. 6 Total time vs half rotation angle.

$$a = R_a(1 + \sin^2\phi)/2 = R \quad (58)$$

Since the semimajor axis is equal to the radius, the transfer ellipse is entered at its semiminor axis. The time required for these transfers is given by the order $R_a^{3/2}$ term of Eqs. (40–42).

The transfer ellipse of this approximate solution is compared with the transfer ellipse of an exact numerical solution in Fig. 3. The numerical solution was found by minimizing the cost function $\Delta V(t/\mu)^{1/3}$ by an accelerated gradient method. The figure is drawn for the maximum rotation angle that can occur with circular initial orbits. For smaller rotation angles the differences between the exact and approximate solutions will be smaller.

Figure 4 illustrates the difference in the cost function between the exact and approximate solution. For values of ϕ up to 45°, the difference in cost is negligible. For larger values of ϕ which become of interest for elliptic initial orbits, the difference in cost is still quite small, reaching a maximum of 2.3% at $\phi = 90°$. As this difference occurs in a first-order correction term, it would usually represent a negligible portion of the zero order ΔV.

For a given apoapsis radius of the unit eccentricity ellipses the approximate solution has a lower ΔV and a longer transfer time. Figure 3 is drawn for the same apoapsis radius rather than the same transfer time. The ΔV requirements of the two solutions is illustrated in Fig. 5. The ΔV of the approximate analytic solution is within 0.4% of the absolute minimum ΔV two impulse transfer between unit eccentricity ellipses. The latter transfer has not been treated in detail because it requires longer transfer times than the approximate analytic solution.

The time for the two transfer modes is illustrated in Fig. 6. The approximate solution takes somewhat longer for large rotation angles because the radial velocity after the impulse is always positive. The numerical solution has negative radial velocities after the impulse for values of ϕ in the vicinity of 90°. Figures 7 and 8 show the radii of the impulses and the angle of the impulses with the horizontal for the two solutions. For the approximate solution, the impulse is always circumferential.

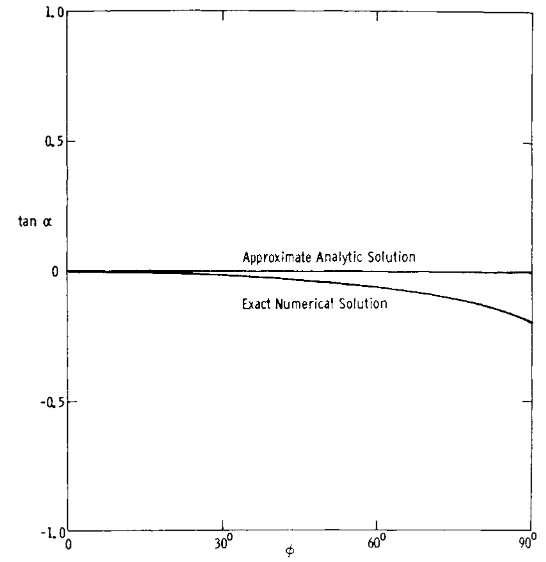

Fig. 8 Tangent of thrust angle vs half rotation angle.

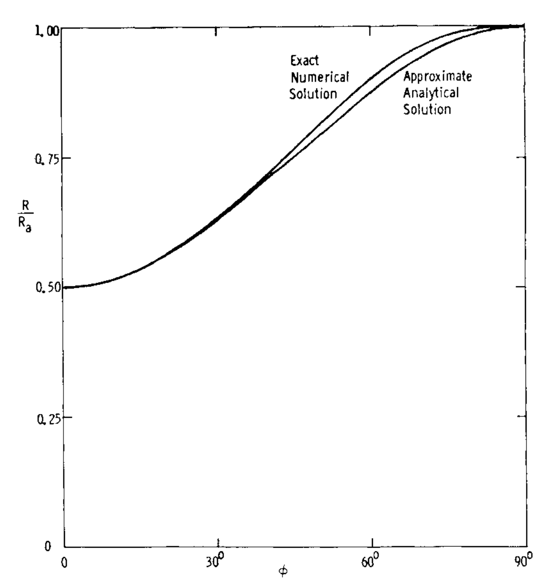

Fig. 7 Radius of impulses vs half rotation.

References

[1] Gerbracht, R. J. and Penzo, P. A., "Optimum Three-Impulse Transfer between an Elliptic Orbit and a Noncoplanar Escape Asymptote," AAS Paper 68-084, Jackson, Wyo., 1968.

[2] Gobetz, F. W. and Doll, J. R., "A Survey of Impulsive Trajectories," *AIAA Journal*, Vol. 7, No. 5, May 1969, pp. 801–834.

[3] Battin, R. H., "Two-Body Orbital Transfer," *Astronautical Guidance*, McGraw-Hill, New York, 1964, Chap. 4, pp. 93–133.

[4] Gunther, P., "Asymptotically Optimum Two-Impulse Transfer from Lunar Orbit," *AIAA Journal*, Vol. 4, No. 2, Feb. 1966, pp. 346–352.

[5] Deerwester, J. M., McLaughlin, J. F., and Wolfe, J. F., "Earth Departure Plane Change and Launch Window Considerations for Interplanetary Missions," *Journal of Spacecraft and Rockets*, Vol. 3, No. 2, Feb. 1966, pp. 169–174.

[6] Webb, E. D., "Three-Impulse Transfer from Lunar Orbits," Paper 66-134, July 1966, American Astronautical Society.

[7] Jezewski, D. J. and Rozendaal, H. L., "An Efficient Method of Calculating Optimal Free-Space N-Impulse Trajectories," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2160–2165.

[8] Bean, W. C., "Minimum ΔV, Three-Impulse Transfer onto a Trans-Mars Asymptotic Velocity Vector," TN MSC S-231, Feb. 1970, NASA.

[9] Bossart, K. J., "Techniques for Departure and Return in Interplanetary Flight," *Aerospace Engineering*, Vol. 17, No. 10, Oct. 1958, pp. 44–52.

[10] Edelbaum, T. N., "How Many Impulses," *Astronautics and Aeronautics*, Nov. 1967, pp. 64–52.

[11] Edelbaum, T. N. and Pines, S., "Fifth and Sixth Integrals for Optimum Rocket Trajectories in a Central Field," *AIAA Journal*, Vol. 8, No. 7, July 1970, pp. 1201–1204.